Serie 12

Optimal transport, Fall semester

EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 12.1. Let $E \in C^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$ be such that the function $E(x) - \frac{\lambda}{2}||x||^2$ is convex. Show that a C^1 function u_t is a solution of $\partial_t u_t = -\nabla E(u_t)$ if and only if the evolutionary variational inequality (EVI)

$$\frac{1}{2}\frac{d}{dt}||u_t - v||^2 + \frac{\lambda}{2}||u_t - v||^2 + E(u_t) \le E(v) \tag{1}$$

holds for any $v \in \mathbb{R}^d$.

Suppose now that we have two curves u_t and v_t satisfying $\partial_t u_t = -\nabla E(u_t)$. Prove that, if we define $d(t) = \|u_t - v_t\|^2$, then

$$d(t) \le d(0)e^{-\lambda t}.$$

In particular, if $\lambda > 0$ and w_0 is the unique minimizer of E, then $\|u_t - w_0\|^2 \le 2(\|u_0\|^2 + \|w_0\|^2)e^{-\lambda t}$.

Exercise 12.2. Recall the Benamou-Brenier formula: given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, then it holds that

$$W_2^2(\mu_0, \mu_1) = \inf\left\{\int_0^1 \int_{\mathbb{R}^d} |v_t|^2 d\rho_t\, dt : \partial_t \rho_t + \operatorname{div}(v_t \rho_t) = 0, \rho_0 = \mu_0, \rho_1 = \mu_1\right\}.$$

Suppose that μ_t for $t \in [0,1]$ is a curve attaining the minimum, and suppose that $\mu_t = (X_t)_{\#}\mu_0$, for some smooth vector field X_t. Prove that $\ddot{X}_t \equiv 0$ μ_0-a.e. for a.e. $t \in (0,1)$.

Exercise 12.3. Let $\mu_0 = \rho_0 \mathscr{L}^d, \mu_1 = \rho_1 \mathscr{L}^d \in \mathcal{P}(\mathbb{T}^d)$ be two probability measures on the d-dimensional torus such that $\rho_0, \rho_1 \ge c > 0$ everywhere. Let $u : \mathbb{T}^d \to \mathbb{R}$ be a solution of the Poisson equation $-\Delta u = \rho_1 - \rho_0$. Show that

$$W_2(\mu_0, \mu_1) \le c^{-1/2}\|\nabla u\|_{L^2}.$$

Hint: Use the Benamou-Brenier formula, which is valid also on the torus.

Exercise 12.4. Let $U : [0,\infty) \to \mathbb{R}$ be a convex function with $U(0) = 0$ such that the energy functional $\mathscr{F}(\rho) := \int_{\mathbb{R}^d} U(\rho)dx$ is W_2-convex. Prove that the function $(0,\infty) \ni s \mapsto U(1/s^d)s^d$ is non-increasing and convex.